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Progress is reported on a study of a new turbulence simulation technique based on an unaveraged, additive decomposition of the Navier-Stokes equations. The decomposition provides a natural separation of the governing equations into large- and small-scale parts, with the small scale solved on local subdomains to provide a high degree of parallelism. Results presented here include formal consistency and accuracy proofs for Burgers' and the full 3-D, incompressible Navier-Stokes equations, as well as various details of transferring information between the large- and small-scale calculations. Initial work on applying the method to the study of transition to turbulence in circular pipe flow is also documented. In addition, studies on the domain decomposition and multigrid aspects of the method, using the Schwarz alternating method and the full-approximation scheme, is included. Keywords:

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AN ADDITIVE TURBULENT DECOMPOSITION OF THE NAVIER-STOKES EQUATIONS IMPLEMENTED ON HIGHLY PARALLEL COMPUTER SYSTEMS
(Final Report)

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AN ADDITIVE TURBULENT DECOMPOSITION OF THE NAVIER-STOKES EQUATIONS IMPLEMENTED ON HIGHLY PARALLEL COMPUTER SYSTEMS

(Final Report)

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1. INTRODUCTION

This report provides a summary of the work performed at UCLA under AFOSR Grant 89-0281, "An Additive Turbulent Decomposition of the Navier-Stokes Equations Implemented on Highly Parallel Computer Systems." A quite detailed report of the first six months of effort was submitted in the form of an Annual Report at the end of October, 1989. The research reported therein will be only briefly summarized here, and the main items treated will represent work completed during the period 1 Jan 1990 to 31 Mar 1990. In the main body of this report the discussions will be mostly qualitative, but an appendix is provided to report the details of pertinent analyses.

The ideas embodied in the additive turbulent decomposition (ATD) were first proposed by McDonough, et al. [1,2] and developed further by McDonough and Bywater [3-5]. The goal was to provide a turbulence simulation technique similar to large eddy simulation (LES), but without the well known deficiencies. In particular, ATD employs no formal filtering of the large-scale equations or modeling on the subgrid (or small) scale. As a consequence, the method can be shown to be formally consistent with the full Navier-Stokes (N.-S.) equations, and thus completely predictive. In this respect, ATD is more closely related to direct numerical simulation (DNS) than to LES; but the decomposition, itself, is reminiscent of LES.

In Refs. 1 through 5 ATD was developed in a rather heuristic manner in the context of Burgers' equation model problems. It has been the goal of the present research to put ATD on a stronger, more formal, mathematical basis, and at the same time extend its application to the full N.-S. equations. In particular, the two main tasks of the proposed research were: 1) theoretical/computational analysis of ATD for Burgers' equation, and 2) sequential mainframe implementation of ATD for the 3-D N.-S. equations. Both of these tasks include numerous subtasks, as listed in the original proposal, McDonough, et al. [6], and in the earlier progress report, McDonough, et al. [7].

In this final report we summarize the work completed on these tasks. In Sec. 2 we

describe the studies conducted on Burgers' equation; Sec. 3 contains an analysis of the work with the N.-S. equations, and in Sec. 4 we provide a summary of the work on domain decomposition, which was a major subtask of the proposed Burgers' equation research. Finally, in Appendix A we report the details of the domain decomposition work.

2. ANALYSIS OF BURGERS' EQUATION

The equation that has been studied in the present work is of the form

$$U_t + 1/2(U^2)_x - \frac{1}{Re} U_{xx} = -P_x . \tag{1}$$

This equation is analogous in form to the incompressible N.-S. equations, except that P_x is a prescribed forcing term, rather than an unknown to be determined so as to enforce mass conservation. During the study reported here, four main projects were completed using this equation. These are: 1) formal proof of consistency of the additive decomposition, and construction of a predictor-corrector algorithm that maintains full order accuracy, 2) development of an energy conserving scheme for transferring large-scale information to local small-scale equations, 3) derivation of an iterative scheme for guaranteeing continuous recoupling of local small-scale results to obtain a global small-scale solution, and 4) establishing a unique value for the decomposition parameter, denoted as β in [1-5]. We note that this last result was actually first obtained for the full N.-S. equations, and so will be discussed more in Sec. 3. These four results enable one to prove formal consistency of the ATD algorithm with a direct solution of (1), thus providing the desired mathematical foundation. An additional result, not directly related to ATD, per se, but still of significance, was also obtained during studies of the previous year. An analysis was undertaken to attempt to assess the effects of various averaging procedures leading to the Reynolds averaged equations, but again in the simple context of Burgers' equation (1). The results obtained

from this show that time averaging is fundamentally flawed, while errors can also be introduced by ensemble averaging if this is done in an improper manner. We will provide further brief discussion of each of these topics in the subsections that follow.

2.1 Consistency and Accuracy of ATD

To state the main result of this analysis we need the following notations. We assume the solution U(x,t) if (1) can be decomposed as

$$U(x,t) = u(x,t) + u^*(x,t)\,, \tag{2}$$

and similarly, the forcing function is represented as

$$P_x(x,t) = p_x(x,t) + p_x^*(x,t)\,, \tag{3}$$

where u, p_x can be viewed as consisting of the first few terms in a Fourier expansion (and hence correspond to low frequency, large-scale behavior), and the "*"-quantities represent the high frequency, small-scale remainders of the series.

We now substitute (2) and (3) into (1), and additively decompose the result to obtain

$$u_t + 1/2(u^2)_x + (1-\beta)(u^*u)_x - \frac{1}{Re}\,u_{xx} = -p_x\,, \qquad \text{(large-scale)} \tag{4a}$$

$$u_t^* + 1/2(u^{*2})_x + \beta(uu^*)_x - \frac{1}{Re}\,u_{xx}^* = -p_x^*\,. \qquad \text{(small-scale)} \tag{4b}$$

We note that the value of the decomposition parameter β is arbitrary with respect to the mathematical validity of the decomposition itself. But it can be shown via physical arguments for the N.-S. equations that $\beta = 1/2$ should be required. This can also be argued on the basis of applying renormalization group theory to transfer information from the small scale to the large scale. In particular, when $\beta = 1/2$ Eqs. (4a) and (4b) are identical in form. It should also be observed that there is no "closure" problem associated with Eqs. (4); there is

one equation for each unknown.

We can now state the main consistency and accuracy result. This will be done in the context of the following algorithm.

Algorithm

Suppose n time steps have been computed for u and u in Eqs. (4).*

Then the n+1th step is obtained from the following procedure:

1) calculate a p-1th order prediction, $\tilde{u}$, of u^{n+1} using (4a);

2) use $\tilde{u}$ in (4b) to solve for $\tilde{u}^{*,n+1}$ via a pth order method;

3) correct $\tilde{u}$ to obtain u^{n+1} using $\tilde{u}^{*,n+1}$ in a pth order method for Eq. (4a).

The following theorem is proven in [7].

Theorem. Let U(x,t) be a globally pth order approximation to the solution of Eq. (1), and suppose the above algorithm is used to calculate u(x,t) and u(x,t) from Eqs. (4). Then u + u* = U + O(k^p) , where k is the time step used to integrate (4a).*

2.2 Transfer of Information between Large and Small Scales

It is important to note that Eq. (4b) is actually solved on numerous local subdomains via a Galerkin procedure for each separate one. This leads directly to nearly an order of magnitude reduction in total arithmetic, and at the same time to the potential for massive parallelization. However, the local spectral representations must incorporate large-scale, global spectral information, and this must be accomplished so as to conserve energy.

We let $\phi_N(x)$ denote the highest basis function of the representation supported on the large scale, and let Ω_i denote the ith small-scale subinterval such that $\bigcup \Omega_i = [0,1]$. It is proven in [7] that the correct lowest mode coefficient on the small scale is given by

$$a_l^{(i)} = \int_{\Omega_i} u\, \phi_N \, dx \, . \tag{5}$$

This is easily approximated via numerical quadrature.

2.3 Global Continuity of Small-Scale Solutions

There are several ways by which the local small-scale solutions can be recoupled to achieve a required degree of global continuity. The approach outlined here has been used by other researchers in different applications. In [7] we provided a fairly detailed derivation and stated that the result would be a global $u^* \in C^0(0,1)$. (In fact, we can show by a different analysis that $u^* \in C^1(0,1)$ holds.) The basic idea is to expand the local representations of u^* by additional basis functions, and then determine the Fourier coefficients of these so as to obtain matching of solutions across subdomain boundaries. On the subdomain Ω_i we have the two additional coefficients

$$a_{K+1}^{(i)}(t) = A_i(t) \quad \text{and} \quad a_{K+2}^{(i)}(t) = B_i(t) - A_i(t) \, , \tag{6}$$

where the A_i's and B_i's are unknown. Now A_1 will be chosen to be consistent with Dirichlet conditions at $x = 0$, and similarly for B_N at $x = 1$. The general matching condition to guarantee global continuity of u^* is

$$A_{i+1}(t) = B_i(t) \, . \tag{7}$$

Typical implementations enforce this iteratively.

It is worth mentioning here that recent work on this problem indicates that a better treatment may be to transform all subdomain boundary information to the local differential operators before spectral discretization. Then the subdomain matching information can be obtained directly as the solution to a tridiagonal linear system for either the A_i's or B_i's.

Moreover, it is much easier to maintain orthonormality of subdomain basis functions with this approach.

2.4 Analysis of Averaging Techniques Via Burgers' Equation

This particular topic is not strictly a part of the research involving the ATD algorithm. It was, however, motivated by two aspects of that work. First, because there are numerical reasons for employing filters in LES, it was desirable to study averaging, in general, to understand its consequences. Second, although it is expected that ATD will significantly expand the Re-range in which accurate turbulence simulations can be done, at present it is clear that most problems still will not admit anything near to DNS. In such cases, ATD may be useful as a research tool in the same sense as is LES and DNS. Hence, it is important to understand just how ATD results should be processed.

The basic idea was to use a direct simulation of Burgers' equation (1) to provide essentially exact Reynolds stresses for use in Reynolds averaged versions of Burgers' equation via the definition

$$u'(x,t) = U(x,t) - \overline{U}\,, \tag{8}$$

where " – " denotes the same averaging used to derive the Reynolds averaged equations, and " ' " denotes a fluctuating quantity. Our findings in this study were reported in [7], and in more detail by Peng [8].

We will simply list the principal results here. First, if time averaging is employed, we let $\bar{u}(x)$ denote the solution to the Reynolds averaged Burgers' equation with Reynolds stress constructed from (8); then numerical experiments show that $\overline{U}(x,t) \neq \bar{u}(x)$. Second, if ensemble averaging is used equality does hold. This can be shown both analytically, and through numerical experiments. Third, again using ensemble averaging, but in place of (8) we use the very natural although inconsistent construction,

$$u'(x,t) = U(x,t) - u(x)\,, \tag{9}$$

where u(x) is a known steady flow that is perturbed, say by a grid or a vibrating ribbon, to produce U(x,t), then $\langle U(x,t)\rangle \neq \langle u(x,t)\rangle$ where $\langle \cdot \rangle$ denotes ensemble averaging.

These negative, and dismaying, results are important for two reasons. First, with respect to implementation of ATD, and use of results thereof, they show that the equations, themselves, must not be averaged. If it proves necssary to provide a large-scale numerical cut-off in some problems, the above results show that the correct way to do this is to compute the large-scale solution, and then average (or filter) the result. Second, we feel that the basic failure of the averaging process, itself, separate from any Reynolds stress modeling, provides a clue as to why classical turbulence closures seem never to be robust. In particular, all effort has been concentrated in the Reynolds stress modeling; but the Reynolds averaged equations, themselves, are evidently incorrect if time averaging has been employed (which is nearly always the case), and explicit account of this has never been taken.

3. ATD APPLIED TO THE N.-S. EQUATIONS

In this section we will discuss the work accomplished up to this time on the full 3-D N.-S. equations,

$$\nabla\cdot\mathbf{U} = \mathbf{0}\,, \tag{10a}$$

$$\mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} - \frac{1}{\mathrm{Re}}\,\Delta\mathbf{U} = -\nabla P\,. \tag{10b}$$

In these equations $\mathbf{U} \equiv (U,V,W)^T$ is the velocity vector, and P is the pressure. The main results that have been obtained are the following: 1) extension of the consistency and accuracy theorem proven for Burgers' equation, 2) identification of a unique decomposition parameter (tensor, in the N.-S. case) via momentum transport considerations, 3) literature

review for the problem of transition in pipe flow, 4) formal application of ATD to the cylindrical coordinate N.-S. equations, and 5) analysis of the corresponding small-scale equations. The details of these studies are quite lengthy and have been reported in [7]; hence, they will not be repeated here. But items 1) and 2) are particularly significant from a theoretical standpoint, so a brief discussion will be provided.

As would be expected from the form of Burgers' equation, the consistency and accuracy results discussed in Sec. 2 can be proven for the N.-S. equations (10). In fact the same line of reasoning given in [7] carries through. This is a crucial theoretical result because it shows that very efficient algorithms can be used to implement the ATD procedure. There are, of course, other alternatives as will be discussed below in Sec. 4, and in Appendix A.

The identification of a unique decomposition parameter is also of great significance. In particular, the apparent nonuniqueness of the decomposition procedure was one of the shortcomings discussed in [1-5], and there was early concern that this might lead to a closure problem. But the analysis presented in [7] shows that the N.-S. decomposition tensor β should simply be the identity tensor I. This results from very simple physical arguments involving momentum transport, and it leads to a decomposition for which the large- and small-scale equations are identical in form. As mentioned earlier, this permits application of renormalization group techniques to transfer results from small to large scale, and it also gives a unique value of $\beta = 1/2$ for the Burgers' equation decomposition parameter.

Our remaining discussions in this section concern work completed on the small-scale equations between 1 Jan 1990 and 31 Mar 1990. In our previous report [7], we defined a series of spectral basis functions, based on the Legendre polynomials, and identified a non-iterative splitting procedure for integrating the small-scale equations. The basis functions were defined such that all but two of the functions in a basis set satisfied homogeneous Dirichlet boundary conditions, while the remaining two in combination satisfied arbitrary inhomogeneous Dirichlet boundary conditions. These functions give rise to a method which

is a hybrid of a Galerkin method (in which all the basis functions satisfy the boundary conditions) and a tau method (in which the coefficients of the highest two modes are chosen after the rest of the coefficients are evaluated, such that the boundary conditions are satisfied). The splitting method we proposed required only Dirichlet conditions on the velocity, which could be easily met by the functions in the basis set.

The basis functions and integration method were selected with an eye to the efficiency of the overall algorithm. A potentially time-consuming operation is the matching of the small-scale solutions at the boundaries between adjacent cells, since this requires information on the small-scale boundary conditions throughout the global (large-scale) domain. The inhomogeneous boundary information was restricted to just two functions in the basis set in order to minimize the arithmetic involved in matching the boundary solutions. Similarly, because the small-scale equations must be solved repeatedly in each of many small-scale cells, we chose the non-iterative splitting method for the integration in order to avoid the extra arithmetic of an iterative method. Note, however, that even though the splitting method is nominally non-iterative, some iteration is required for the accurate evaluation of the non-linear terms.

Further examination of the small-scale problem, with the continuing goal of minimizing the overall arithmetic, led us to a re-evaluation of the basis functions and integration algorithm initially selected. This resulted in the identification of a more efficient integration algorithm and a different series of basis functions. These are described below, along with the motivation for their selection.

Our reconsideration of the basis functions was prompted by our desire to minimize the arithmetic associated with the evaluation of the nonlinear terms in the momentum equation. The evaluation of the (nonlinear) quadratic terms in a spectral representation of the Navier-Stokes equations must be carried out via the process of convolution. This is a crucial step, and the efficiency of the algorithm may depend on the efficiency with which the convolutions are evaluated. The most efficient methods for the evaluation of spectral

convolutions make use of so-called "fast" spectral transforms. Well-known fast transforms exist for the trigonometric (Fourier) polynomials, and for the Chebychev polynomials. A fast transform also exists for the Legendre polynomials [9], but it is more difficult and less well known than the Fast Fourier Transform (FFT). Furthermore, all these fast transforms manipulate the coefficients of the weighted sums of the individual spectral functions on which the transform is based. Additional manipulations -- and additional arithmetic -- are necessary if a Galerkin or Galerkin-like method has been used in the spectral discretization of the original differential equation, i.e., when the spectral functions are used in combinations such as those we had earlier proposed, rather than being used individually.

These facts led us to consider the use of Chebychev polynomials and a pure tau method. The use of a Chebychev tau method permits the convolutions to be carried out by means of Fast Chebychev Transforms (FCT's), without the additional manipulations that a Galerkin method would require. This requires O((N–2)log(N)) arithmetic in two dimensions, and O((N–3)log(N)) in three. We then looked for a method of solving the equations as a whole which demanded a comparable level of arithmetic.

In Section 5.1.3 of their book, Canuto, Hussaini, Quarteroni, and Zang [10] give several efficient methods for the Chebychev tau solution of Poisson equations. We can use these methods provided that we can manipulate the discrete Navier-Stokes equations into Poisson form. The derivation and use of a pressure Poisson equation to replace the mass conservation equation is well known, but due care must be exercised in the imposition of the boundary conditions (cf. Gresho and Sani [11]). It is also possible to put the discrete momentum equations in Poisson form if we use implicit Crank-Nicolson to time-discretize the linear terms, but use a predictor-corrector approach in the time-discretization of the nonlinear and pressure gradient terms. An explicit treatment of these terms would also work, but a predictor-corrector approach is preferred to a simple explicit treatment because the corrector can be iterated an arbitrary number of times in order to improve the stability of the integration. With the equations in this form, the matrix diagonalization approach described

by Canuto, et al. [10], may be used to solve the entire system in O(N) arithmetic in both two and three dimensions. In this way, the solution of the system of equations is made almost as efficient as the evaluation of the convolutions. We also note that the use of a predictor-corrector approach extends straightforwardly to multi-step methods of higher time-accuracy.

As noted above, we had initially considered our composite Legendre-based basis functions in order to facilitate the matching of the small-scale solutions at the subdomain (cell) boundaries. The adoption of the Poisson equation approach, however, also facilitates this operation by opening up the extensive literature on the solution of the Poisson equation by means of domain decomposition, which necessarily considers the problem of solution-matching at the subdomain boundaries. Additionally, the boundary conditions required by the pressure Poisson equation [10] suggest that C^2-continuity of the velocity should automatically guarantee C^1-continuity of the pressure, and in light of the pressure Poisson equation this should guarantee global mass conservation. We shall be examining this further in terms of its relation to the theory of domain decomposition, the requirements for accurate matching of the small-scale solutions, and the connection between the small and large scales.

In summary, therefore, the new approach shares with the original method the characteristics of facilitating the matching of solutions at the small-scale subdomain boundaries and using an efficient routine for the small-scale integrations, but in addition it further minimizes the overall arithmetic and provides a stronger connection to the theory of domain decomposition.

We have defined a test problem for the small-scale algorithm: it is a single 2-D small-scale cell with C^1-periodic boundary conditions, and with an imposed constant large-scale shear. This is a single-parameter problem, with the one arbitrary parameter being the Reynolds number defined in terms of the cell height and the large-scale velocity gradient. This problem will be run at a number of different Reynolds numbers in order to map out the parameter space of qualitative behavior of solutions in a treatment similar to that of McDonough and Bywater [3,4]. Coding of this problem is in progress. Because of our

choice of algorithms, this can be directly extended to the 3-D case, and in this context will simply be a callable subroutine in the overall small-scale solution process.

4. DOMAIN DECOMPOSITION

There has been very significant progress in the domain decomposition work during the period 1 Jan 1990 to 31 Mar 1990. We will briefly describe the main parts of this in the present section and leave details to Appendix A. The viewpoint taken is somewhat different from that of the work of McDonough and Bywater [3-5]. In particular, ATD will here be analyzed in the context of a two-grid multigrid method with the large-scale corresponding to the coarse grid, and the small scale associated with fine grid calculations. Because of the nonlinearity of Burgers' and the N.-S. equations, we have employed the so-called full approximation scheme (FAS) in the present work. It is shown in Appendix A that with this approach, the differential equation(s) remains the same on all scales (as with standard ATD) but requires a forcing term that is different for different scales.

Two particular implementations employing domain decomposition have been studied for Burgers' equation: 1) the Schwarz alternating method, and 2) a Gauss-Seidel-Newton method. These have been run in a sequential manner to date, but both are easily parallelized. We now list the main results. First, for the Schwarz alternating method it was found that convergence of the small-scale recoupling iterations (to the level of discretization error) could be achieved in a single iteration provided a sufficient number of modes was carried on each of the small-scale subdomains. Conversely, if too few modes were used, convergence occurred only to a certain level (determined by the discretization error). Moreover, it was found that the coarse grid results were not even needed for the model problems. It should be noted that this last result is not expected to hold for a real physical flow field simulation. On the other hand, the remaining observations show that this approach holds much promise.

Results for the Gauss-Seidel-Newton studies indicate that this approach is not as robust

as is the Schwarz method. Its performance is somewhat more sensitive to time step size, and in general more iterations are required per time step. But the arithmetic per iteration is much less than for the Schwarz method.

Further details of these studies, and the conclusions that can be drawn from them, are provided in Appendix A.

5. SUMMARY

In summarizing the work on this project we note that all of the proposed work for Burgers' equation has been completed. In addition, another major project not originally proposed was undertaken and carried to conclusion. One Masters Thesis, Peng [8], and one conference presentation, Peng and McDonough [12], have resulted from this part of the research; also, two journal articles are in progress.

The work associated with the 3-D N.-S. ATD implementations is not yet complete, but good progress has been made particularly with regard to theoretical aspects. One conference presentation has been made, Hylin and McDonough [13], on this portion of the research. Furthermore, the work has proceeded continuously beyond the 31 Mar 1990 project ending date.

The domain decomposition studies, although begun somewhat later than hoped, have by now produced significant useful results. As mentioned in Sec. 4, these have been along somewhat different lines than had originally been expected, but are nevertheless valuable in understanding the best approaches to follow in ATD implementations.

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APPENDIX A: Domain Decomposition Studies

REPORT ON

Studies of the Additive Turbulent Decomposition Method.

by Tony F. Chan and Tarek Mathew

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1 Introduction.

The Additive Turbulent Decomposition method (ATD) is a numerical technique under study and development for the purpose of computing qualitatively accurate approximations to turbulent solutions of the incompressible Navier-Stokes equations. It is described in McDonough and Bywater [22], [23]. In this document, we report on our studies of the domain decomposition aspects of the ATD method. This was done during the grant period this year (until March 1990).

The rest of this document is outlined as follows. First, we briefly describe the original version of the ATD method. This involves the use of a coarse grid model and a fine grid model for the discretisation of the problem. We then describe the framework in which we have studied the ATD method, namely as a two level multigrid-domain decomposition type method. We also include a discussion of the work by Henshaw, Kreiss and Reyna [17] on the smallest scale for the incompressible Navier-Stokes equations which has some implications on the selection of the discretisation used in the ATD method. The original ATD method involved a coupling between the coarse grid and fine grid model equations. Our studies have led us to incorporate another technique to couple these equations, namely, the *Full Approximation Scheme* (which is frequently used in nonlinear multigrid methods). We describe this technique and also some domain decomposition algorithms that are used in conjunction with this. We then present numerical results of tests done using these methods on a model problem. Following that we discuss briefly the extensions of these methods to the case of the Stokes problem and incompressible Navier-Stokes equations. Finally, we describe some work we plan to do next year.

2 The ATD method.

Here we briefly recall the original version of the ATD method of McDonough and Bywater [22], [23]. Consider an evolution equation:

$$U_t + L(U) = F, \tag{1}$$

for a nonlinear elliptic operator $L(.)$, forcing F, and appropriate boundary conditions and initial data. The solution U is decomposed, in some suitable way, as the sum of a *large-scale* component U_l (representing the large-scale features of the solution) and a *small-scale* component U_s (representing the small-scale, local features of the solution) evolving on a faster time scale:

$$U = U_l + U_s.$$

Using this, we can rewrite (1) as a pair of equations for U_s and U_l:

$$\begin{aligned} \frac{\partial U_s}{\partial t} + L_1(U_s + U_l) &= F_s \\ \frac{\partial U_l}{\partial t} + L_2(U_s + U_l) &= F_l \end{aligned} \tag{2}$$

where $L_1(.)$ and $L_2(.)$ are nonlinear operators that sum to $L(.)$ (see [22], [23] for their definition), and $F = F_s + F_l$. The equation for the *small-scale* component U_s is posed locally on a partition of the domain and is discretised locally on each of the subregions using a suitable method. The equation for the *large-scale* component U_l is discretised on a coarse grid. Since the two equations in (2) are coupled, an iterative procedure must be used to solve them, e.g., by alternatingly freezing one component while updating the other. Of course, the splitting $L = L_1 + L_2$ must be chosen so that we obtain a convergent method and so that the interaction between the large-scale and small-scale components of the solution are well represented. On this issue, we were led to consider a frequently used procedure in nonlinear multigrid methods, known as the *full approximation scheme*. We describe this in a later section.

3 A framework for our studies.

The ATD method represents a remodelling of the discretisation process of the differential equations, as well as a divide and conquer type strategy to compute the solution of the discrete equations. The discretisation involves two grids, a fine grid

and a coarse grid, and the iteration involves the solution of subproblems on subdomains. In the case of direct simulations of turbulence, the fine grid problem is solved exactly, to obtain the fine grid solution U_h, where h denotes the size of the fine grid. The choice of h must be based on the smallest scale of the flow: $h \leq h_{min}$, which is discussed in Henshaw, Kreiss and Reyna [17]. There they prove that the minimum scale associated with two and three dimensional turbulent flows of the incompressible Navier-Stokes equations, is inversely proportional to the square root of the Reynolds number based on the kinematic viscosity and the maximum of the velocity gradients. Thus, for large Reynolds numbers, the solution of the fine grid problem may be prohibitively expensive. Furthermore, solving a coarse model problem for U_H, though of moderate expense, may not be accurate as mentioned above. In the context of other methods to simulate or model turbulence, such as Large Eddy Simulation (LES), see [10], [11], the ATD method was initiated with the goal of computing an approximation $U_{h,H}$ to the fine grid solution U_h, based on the fine grid, at a cost considerably less than direct simulation, and more accurate than the coarse grid solution U_H.

These considerations led us to formulate the ATD method in the framework of a two-level multigrid method, in which the *smoothing* (a procedure which efficiently captures the small-scale or high frequency local features of the solution) is done by a domain decomposition method. And in accordance with our framework, we will use terms such as *small-scale* and fine grid interchangeably, as well as *large-scale* and coarse grid.

However, this point of view needs modification (though we have not focused on this so far), since unlike domain decomposition methods for stationary problems, the presence of two spatial scales which may evolve at different time scales adds an extra structure to the problem. For instance, it is observed that following an initial time range during which the flow is highly random (maximally dissipative), the flow organises itself into coherent structures (mature flows) with some noticeable large-scale features. The large-scale features may evolve at a slower time scale than the small scale features. Thus, to increase the efficiency of a numerical technique simulating turbulence, the technique should be adapted to the regime of the flow. We hope to address this issue in our future studies making use of quantitative bounds for the minimum scale based on maximum velocity gradients (as presented in [17]), to adaptively coarsen or refine the mesh. We also intend to test the use of different time

steps corresponding to different spatial scales in the framework for the ATD method, as such techniques have been used in some multigrid CFD codes for compressible flows, see Jameson, Schmidt and Turkel [18]. We hope to explore this in the context of the incompressible Navier-Stokes equations.

In the case when more explicit information about the nature of the flow is known, such as when the flow is locally homogeneous, then it may be possible to solve exactly for just one of the subregions to form a local subgrid turbulence model and use this information to form a composite small-scale model on the whole domain, with a cost that is considerably reduced. Of course, a coarse model would be needed in addition, to capture the large-scale features of the flow. Such features can be tested numerically, in the corresponding regimes for the flow.

Another issue in constructing a framework for the ATD method, is on the choice of discretisation. Results of numerical experiments, presented in Browning and Kreiss [5], on the simulation of turbulence for the incompressible Navier-Stokes equations, indicate that the use of some lower order finite difference schemes in space with the same number of nodes as a Fourier pseudo-spectral method, integrated in time using a fourth order predictor-corrector scheme, remained accurate for short time ranges but became inaccurate after long time integrations, whereas higher order finite difference schemes and the pseudo-spectral method remained accurate. Such results indicate the importance of the selection of discretisations in the ATD method.

4 The Full Approximation Scheme.

As mentioned earlier, the goal of the ATD method is to efficiently compute an approximation to the fine grid solution U_h based on the coarse grid and fine grid equations. The coupling between the two scales is represented in equation (2). However, an optimal choice for the forcing terms or for the splitting $L = L_1 + L_2$ to couple the two scales, is not clear. This led us to incorporate into the ATD framework a technique commonly used in nonlinear multigrid methods to couple the fine grid and coarse grid problems, namely, the *Full Approximation Scheme* (FAS). The FAS has some of the above desired features, see Brandt [3] and Hackbusch [16]. We briefly describe it here for a stationary problem, though it holds virtually without change for time dependent problems, when an implicit time stepping method is used.

Assume that the nonlinear equation $L(U) = F$ is discretised on two grids, a fine grid $L^h(U^h) = F^h$ and a coarse grid $L^H(U^H) = F^H$. We use I_H^h to denote an interpolation map from the coarse grid to the fine grid, and I_h^H to denote a restriciton map (possibly the adjoint map of I_H^h) from the fine grid to the coarse grid. Given an inexpensive procedure for computing an approximate solution V_h to the fine grid equations, which captures the local features of the solution (by a suitable procedure, often referred to as *smoothing*), we solve for the correction $U^h - V^h$ on the coarse grid, in an indirect way. We solve for an unknown W_{full}^H on the coarse grid by exactly solving

$$L^H(W_{full}^H) = L^H(I_h^H V^h) + I_h^H[F^h - L^h(V^h)].$$

Here W_{full}^H represents an approximation to $I_h^H U^h$, and is therefore called a *full approximation* on the coarse grid. The solution on the fine grid is then updated or improved by modifying it as:

$$V_{new}^h \longleftarrow V^h + I_H^h[W_{full}^H - I_h^H V^h].$$

Following this, the *smoothing* procedure can be repeated once again on the fine grid, to produce an update.

Note that the nonlinear equations remains the same on both the coarse and fine grids, for all iterations, while the forcing changes. Furthermore, heuristic reasons can be given to show that an appropriate forcing for the *large-scale* equations is the one given above, see [3], [16]. Finally, we mention that if the restriction map is chosen to be the orthogonal projection onto the coarse grid, then upon convergence, the *full approximation* W_{full}^H becomes the orthogonal projection of the exact fine grid solution onto the coarse grid, even for nonlinear equations.

5 Solution of small-scale equations.

In this section, we discuss how approximate solutions to the fine grid problem are obtained using two types of *smoothers* (procedures which capture the local or small-scale high frequency features of the solution). We describe a domain decomposition method known as the Schwarz alternating method (and also mention a variant of it for parabolic problems due to Kutznetsov) as well as a nonlinear Gauss-Seidel-Newton method. Both these are, of course, used in conjunction with the FAS on the coarse grid to obtain an approximation to both the small-scale and large-scale features of

the solution. The use of domain decomposition usually leads to easily parallelisable algorithms. In addition, the Schwarz alternating method is also applicable to the Stokes equations (in which case convergence proofs are known, see Lions [20] and Mathew [21]), as well as to the Navier-Stokes equations.

5.1 The Schwarz alternating method.

We now describe the Schwarz alternating method in the case of an elliptic problem. Its generalisation to the case of discrete versions of elliptic problems and to the implicit time discretisations of parabolic problems is immediate. Consider the elliptic problem:

$$L(U) = F \quad \text{in } \Omega,$$

with appropriate boundary conditions on $\partial\Omega$, where $L(.)$ is a nonlinear elliptic operator, and Ω is the domain of the problem. Let Ω be partitioned into ns overlapping subdomains $\{\Omega_j\}$:

$$\Omega = \cup_{j=1}^{ns} \Omega_j.$$

Given an approximation U_i to U satisfying the boundary conditions, we construct a new approximation U^{i+1} in ns fractional steps as follows. For $j = 1$ to ns, solve

$$L(W_j) = F \quad \text{in } \Omega_j,$$

with

$$W_j|_{\partial\Omega_j} \equiv U^{i+\frac{j-1}{ns}}|_{\partial\Omega_j},$$

and define

$$\begin{aligned} U^{i+\frac{j}{ns}} &\equiv W_j && \text{in } \Omega_j, \\ U^{i+\frac{j}{ns}} &\equiv U^{i+\frac{j-1}{ns}} && \text{in } \Omega - \Omega_j. \end{aligned}$$

The method presented above (which is sort of a block Gauss-Seidel method) is the basic version of the Schwarz alternating method, and it is sequential as we have described it. Note that no global linearisations of the nonlinear equations is required. In the case of linear problems, elliptic or parabolic, there is a highly parallelisable version of this algorithm (block Jacobi), which can also be applied to global linearisations of nonlinear problems, see Dryja and Widlund [9], Cai [6].

However, even the Gauss-Seidel version is easily parallelisable via the technique of multicoloring. We briefly describe this with an example. Consider the case where

Ω is the unit square. To obtain $\{\Omega_j\}$, we first partition Ω into a mesh of equally sized subsquares $\{\Omega_j\}$ without overlap. Following that, we enlarge each square to a rectangle so that an overlapping collection $\{\Omega'_j\}$ is formed. Then, the subdomains $\{\Omega'_j\}$ are grouped into four colors, each color consisting of disjoint subdomains on which local problems can be solved in parallel. Thus, the number of sequential steps in this algorithm can be reduced to four. See Figure 1.

For linear elliptic problems, the algorithm can be shown to have a geometric rate of convergence for suitable discretisations, see Lions [20], Dryja and Widlund [9], Bramble, Pasciak, Wang and Xu [2], and Mathew [21], i.e.,

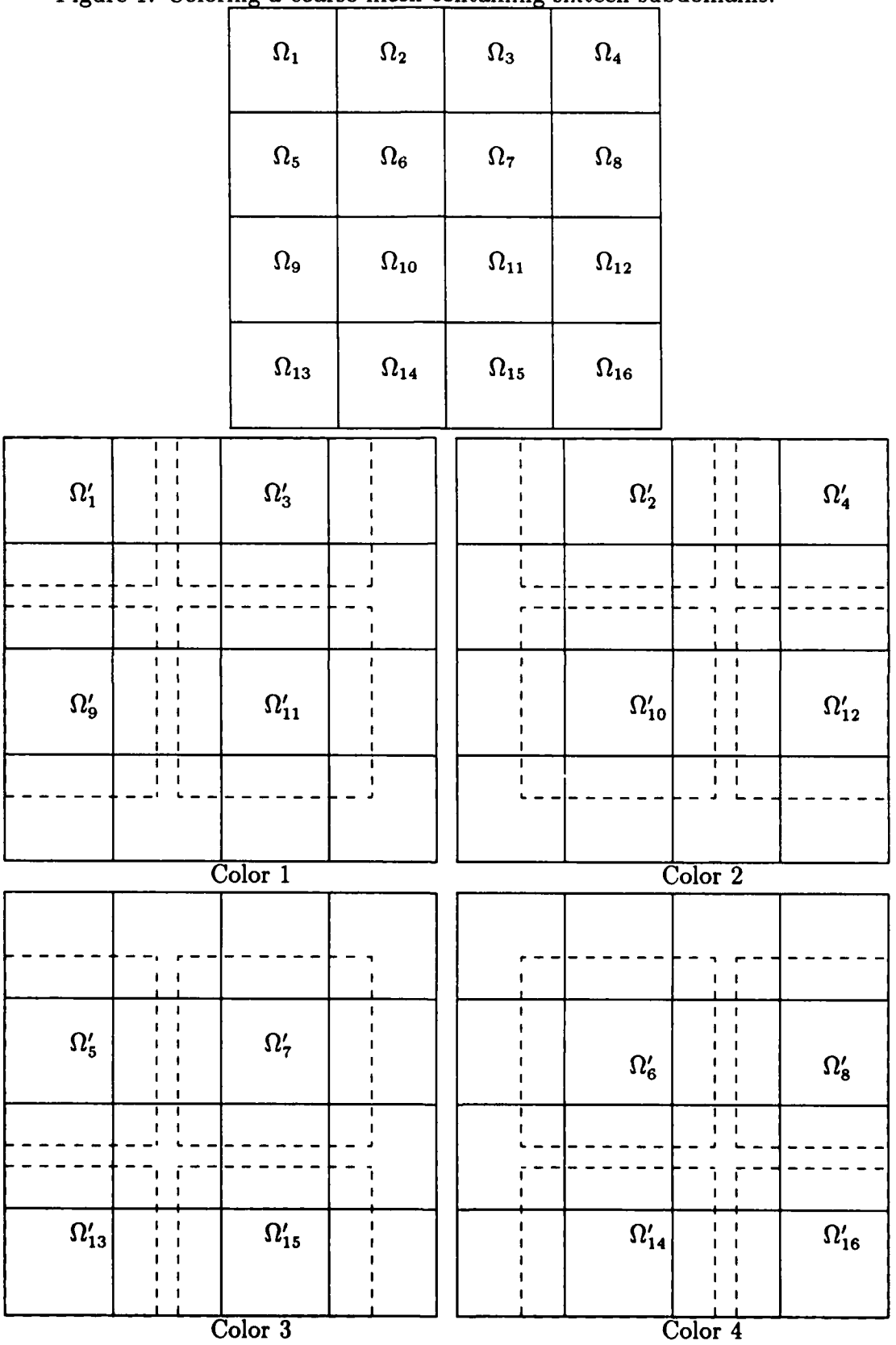
$$\|U - U^{i+1}\| \leq \rho^i \|U - U^0\|,$$

for a constant $\rho < 1$ and independent of h. However, as the number of subdomains ns increases, the rate of convergence of the iteration decreases, due to the global domain of dependency of elliptic problems. This deterioration in the rate of convergence as the number of subdomains is increased, is usually eliminated when the algorithm is used with a coarse grid in the FAS context. Numerical results are presented for a model problem, in the next section.

5.2 Nonlinear Gauss-Seidel Newton method.

Here we briefly describe the Gauss-Seidel-Newton method, which is commonly used as a *smoother* for multigrid methods for nonlinear elliptic problems, see Brandt [3] and Hackbusch [16]. It is, in a way, a limiting case of the Schwarz alternating method, when each subdomain is chosen to contain just one interior unknown. Let $M(U) = F$ denote a system of N nonlinear equations for N real unknowns $U = (u_1, \ldots, u_N)$. Given an approximate solution $V^i \approx U$, we construct the next approximate solution V^{i+1} in N fractional steps. During each fractional step, just one unknown is updated by solving one of the equations in which all other unknowns are frozen. Since each equation is nonlinear, each fractional step may involve the solution of one nonlinear equation in one unknown, say by Newton's method. More generally, just one step of the Newton iteration can be used. This is just a generalisation of the standard Gauss-Seidel method for symmetric positive definite linear systems, to the case of nonlinear systems. An advantage of the Gauss-Seidel-Newton method is that no global linearisation of $M(U)$ is required, rather only linearisations of scalar equations.

Figure 1: Coloring a coarse mesh containing sixteen subdomains.



5.3 An additive Schwarz type method of Kutznetsov.

The time dependent nature of our problem can also be exploited to derive more efficient domain decomposition *smoothers*. Here we briefly describe such an algorithm, requiring no coarse model, for linear parabolic problems, due to Kutznetsov [19], see also Meurant [24]. Of course, this algorithm can also be applied to global linearisations of nonlinear problems. Consider equation (1) for a linear elliptic operator $L(.)$. If an implicit discretisation in time is used, such as backward Euler, the resulting stationary problems have the form:

$$(I + \Delta t L)U^{n+1} = \tilde{F} \equiv U^n + \Delta t F^{n+1}, \tag{3}$$

where Δt is the time step. In the method of Kutznetsov, the domain is subdivided into ns nonoverlapping subdomains $\{\Omega_j\}$, and the forcing $\tilde{F}$ is decomposed as a sum of functions $\{\tilde{F}_j\}$:

$$\tilde{F} = \tilde{F}_1 + \ldots + \tilde{F}_{ns},$$

where $\tilde{F}_j$ has support in Ω_j or a small extension of Ω_j. Following that, given a small positive number ϵ, say on the order of Δt or $(\Delta t)^2$, each subdomain Ω_j is enlarged to Ω'_j by an amount proportional to $\sqrt{\Delta t}\,\log\epsilon$, and ns subproblems are solved in parallel on each subdomain Ω'_j with zero boundary conditions:

$$(I + \Delta t L)W_j = \tilde{F}_j \quad \text{in } \Omega'_j,$$

with $W_j|_{\partial\Omega'_j} = 0$. Then, an approximate solution to equation (3) is obtained by adding the $\{W_j\}$. It can be shown that

$$\|(W_1 + \ldots + W_{ns}) - U^{n+1}\| = O(\epsilon),$$

by using the decay property of the Green's funtion for this problem, see Kutznetsov [19].

5.4 Other methods.

In our studies, we also considered other domain decomposition methods. In particular, we considered the *patching method*, see Canuto, Hussaini, Quarteroni and Zang [7]. However, the technique to patch together the subdomain solutions is quite

complicated, especially for two or three dimensional domains, and so we preferred using the Schwarz alternating method and the nonlinear Gauss-Seidel method instead. More varieties of domain decomposition algorithms exist, see Glowinski et al [13] and Chan et al [8].

6 Numerical tests on a model problem.

In this section, we report on the results of some numerical tests done on a model problem, namely, the viscous Burgers' equation for a large parameter Re.

$$\begin{cases} U_t + UU_x & = & \frac{1}{Re}U_{xx} + f(t,x) & \forall x \in (-1,1); \quad t > 0 \\ U(t=0,x) & = & U_0(x) & \forall x \in (-1,1) \\ U(t,-1) = g_L(t), & & U(t,+1) = g_R(t) & \forall t \geq 0. \end{cases} \tag{4}$$

In our previous notation the above problem would read

$$U_t + L(U) = F,$$

where the nonlinear elliptic operator $L(.)$ is

$$L(U) \equiv \frac{-1}{Re}U_{xx} + UU_x.$$

We discretised Burgers' equation by using backward Euler (and occasionally Crank-Nicolson) in time and Chebychev collocation in space, for each of the subdomains and for the coarse model. In all of the tests, we present the results for the solution of the implicit equations

$$U + \Delta t L(U) = \tilde{F}, \tag{5}$$

which occurs at every time step (similar equation for Crank-Nicolson). This was done to study the efficiency of the various methods for a single time step. Exact solutions were chosen with a sufficiently large number of oscillations.

We compute the relative error using the exact solution to the stationary problem (5). In the tables, we list the number of iterations required to decrease the relative error in the computed solution by a factor $\Delta t/10$ (since we used the backward Euler method which is accurate to order one in Δt).

We present numerical results for:

1. The ATD method based on the Schwarz alternating method (in the context of the FAS), see table 1. Exact solvers were used for the subdomain problems.

2. The ATD method based on the nonlinear Gauss-Seidel-Newton method (in the context of the FAS), see table 2.

3. Kutznetsov's method for a convection-diffusion problem, see table 3.

We now mention some details about the choice of the mesh parameters. For more details, see Gottlieb and Orszag [14], and Canuto, Hussaini, Quarteroni and Zang [7]. The number of modes used in the Chebychev collocation method was based upon stability considerations, since for large Reynolds numbers, $Re \gg 1$, there can be development of boundary layers, and this can cause instability in the spectral discretisation unless the number of modes used, N, is sufficiently large (see Gottlieb and Orszag [14], page 140). The restriction is that $N \geq N_{critical}$, where

$$N_{critical} \approx 3\sqrt{Re \ \|U\|_{max}}.$$

For large Reynolds numbers, the coarse model can become unstable if a small number of modes is chosen. In such cases, we used artifical viscosity, i.e., decreased the Reynolds number on the coarse model, to obtain stable discretisations.

Our choice of the time step was based on the scale of changes of the flow on the computational grid. This was roughly chosen to correspond to the CFL condition for the hyperbolic equation

$$U_t + U_x = 0,$$

which is a model for

$$U_t + UU_x = 0,$$

(which is obtained as $Re \longrightarrow \infty$). For Chebychev collocation methods, this is given by:

$$\Delta t \leq \frac{c}{N^2}, \tag{6}$$

for some positive constant c (which we set as 1 for our tests). In some of our tests, we also used $\Delta t = \frac{1}{N}$ and $\Delta t = \frac{1}{N^{\frac{1}{2}}}$. We contrast this with the time step restriction for explicit schemes:

$$\Delta t \leq \frac{c}{N^4},$$

since the eigenvalues of the discrete Chebychev collocation approximation to elliptic operators in one dimension have modulus on the order of N^4, where N is the number of modes used, see [7].

Table 1: ATD USING SCHWARZ METHOD.

$N = \frac{1}{h}$	N_{coarse}	Δt	Re	SMOOTHINGS	SUBDOMAINS	ITERATIONS
32	8	$1/N^2$	1	4	4	1
32	8	$1/N^2$	1	4	6	1
32	8	$1/N^2$	1	4	8	1
32	8	$1/N^2$	1	4	10	1
32	8	$1/N^2$	1	4	12	1
32	8	$1/N^2$	1	4	14	1
32	8	$1/N^2$	1	4	16	1
32	8	$1/N^2$	10	4	4	1
32	8	$1/N^2$	10	4	6	1
32	8	$1/N^2$	10	4	8	1
32	8	$1/N^2$	10	4	10	1
32	8	$1/N^2$	10	4	12	1
32	8	$1/N^2$	10	4	14	1
32	8	$1/N^2$	10	4	16	1
32	8	$1/N^2$	100	2	8	1
32	8	$1/N^2$	100	4	8	1
32	8	$1/N^2$	100	2	16	1
32	8	$1/N^2$	100	4	16	1
32	8	$1/N^2$	100	2	24	1
32	8	$1/N^2$	100	4	24	1
32	8	$1/N^2$	100	2	32	1
32	8	$1/N^2$	100	4	32	1
128	32	$1/N^2$	1000	2	8	1
128	32	$1/N^2$	1000	4	8	1
128	32	$1/N^2$	1000	2	16	1
128	32	$1/N^2$	1000	4	16	1
128	32	$1/N^2$	1000	2	24	1
128	32	$1/N^2$	1000	4	24	1
128	32	$1/N^2$	1000	2	32	1
128	32	$1/N^2$	1000	4	32	1

Table 2: ATD USING GAUSS-SEIDEL-NEWTON METHOD.

$N = \frac{1}{h}$	N_{coarse}	Δt	Re	SMOOTHINGS	ITERATIONS
32	8	$1/N^2$	1	1	> 10
32	8	$1/N^2$	1	2	6
32	8	$1/N^2$	1	3	4
32	8	$1/N^2$	1	4	4
32	8	$1/N^2$	1	5	3
32	8	$1/N^2$	10	1	5
32	8	$1/N^2$	10	2	3
32	8	$1/N^2$	10	3	2
32	8	$1/N^2$	100	1	3
32	8	$1/N^2$	100	2	2
32	8	$1/N^2$	100	3	1
32	8	$1/N^2$	100	4	1
32	8	$1/N^2$	100	5	1
128	32	$1/N^2$	1000	1	4
128	32	$1/N^2$	1000	2	2
128	32	$1/N^2$	1000	3	2
128	32	$1/N^2$	1000	4	1
128	32	$1/N^2$	1000	5	1
32	8	$1/N$	100	1	5
32	8	$1/N$	100	2	3
32	8	$1/N$	100	3	2
32	8	$1/N$	100	4	2
32	8	$1/N$	100	5	1
128	32	$1/N$	1000	1	∞
128	32	$1/N$	1000	2	∞
128	32	$1/N$	1000	3	∞
128	32	$1/N^{1.5}$	1000	1	6

Table 3: KUTZNETSOV'S METHOD.

$N = \frac{1}{h}$	Δt	Re	OVERLAP N_o	CONVECTION c	ERROR
32	$1/N^2$	1	1	1.0	1.3e-1
32	$1/N^2$	1	5	1.0	1.1e-3
32	$1/N^2$	10	1	1.0	1.0e-2
32	$1/N^2$	10	5	1.0	6.8e-4
32	$1/N^2$	100	1	1.0	4.2e-3
32	$1/N^2$	100	5	1.0	6.3e-4
128	$1/N^2$	1000	1	1.0	2.0e-3
128	$1/N^2$	1000	5	1.0	3.1e-4
32	$0.01/N^2$	10	1	1.0	4.5e-4
32	$0.01/N^2$	10	5	1.0	1.1e-5
32	$0.01/N^2$	100	1	1.0	5.9e-5
32	$0.01/N^2$	100	5	1.0	6.8e-6
128	$0.01/N^2$	1000	1	1.0	3.6e-5
128	$0.01/N^2$	1000	5	1.0	4.0e-6
32	$1/N^2$	100	1	0.5	3.6e-3
32	$1/N^2$	100	5	0.5	3.2e-4
128	$1/N^2$	1000	1	0.5	2.0e-3
128	$1/N^2$	1000	5	0.5	2.2e-4
32	$1/N^2$	100	1	0.0	3.3e-3
32	$1/N^2$	100	5	0.0	8.0e-5
128	$1/N^2$	1000	1	0.0	2.0e-3
128	$1/N^2$	1000	5	0.0	1.8e-4

We now briefly discuss the results. First, we consider the ATD method based on the Schwarz method, see table 1. In this case, our choice of subdomains Ω_j was non-overlapping intervals of equal length. These were then extended halfway into the adjacent subintervals to form the overlapping collection $\{\Omega'_j\}$. The intervals were grouped into two colors, providing an easily parallelisable algorithm. We found that the number of modes used locally, in each subinterval has a critical effect on the convergence of the Schwarz method, i.e., if the number of modes used in each subinterval were below a critical number (depending on the smoothness of the exact solution), then the Schwarz iteration did not reduce the error beyond a certain point. However, if the number of modes used locally were increased, then the method would converge geometrically. This indicated that the discretisation error prevented convergence beyond a certain point, when the number of modes used locally were insufficient. In addition, we have found that for the choice of time steps we used, the Schwarz method converged to within discretisation error in one iteration (provided, the number of modes used locally were sufficient) and that the coarse model was not needed. In table 1 we list the number of iterations required to reduce the relative error of the computed solution by a factor of $\Delta/10$, however, it was observed that the relative error was usually reduced by a factor much smaller than $\Delta/10$. We have also listed the number of *smoothings* used, i.e., the number of times the Schwarz iteration was used before the coarse model correction was done. For the particular choice of time steps, the rate of convergence is independent of the number of subdomains and the Reynolds number. However, as $\Delta t \longrightarrow 1$, the rate of convergence became geometric and was much slower than for smaller time steps.

The next group of tests listed are for the ATD method based on the nonlinear Gauss-Seidel-Newton method, which is a limiting case of the Schwarz method applied to individual unknowns rather than to blocks of unknowns. Each scalar nonlinear equation was solved locally by using a Newton iteration. Here, it was found that for large Reynolds numbers the iterations sometimes diverged when the time steps were not small enough, see table 2. Moreover, as the time step grew bigger, the number of iterations required also increased. This may be related to the lack of diagonal dominance in the resulting system. However, for the time steps $\Delta t = \frac{1}{N^2}$ and $\Delta t = \frac{1}{N^{1.5}}$, the number of iterations required to reduce the initial relative error in the solution by a factor $\Delta t/10$ decreased as the number of *smoothing* iterations

were increased. The computational cost for one iteration of this version of the ATD was considerably less than for the ATD version based on the Schwarz method. This is because the Schwarz based ATD algorithm requires inter-domain communication, and the cost of setting up the local subproblems using the local Chebychev nodes is more expensive than using the Chebychev discretisation on the whole domain.

Our final table 3 contains results of tests done on a model linear convection-diffusion problem:

$$U_t + cU_x = \frac{1}{Re}U_{xx} + f,$$

with appropriate initial and boundary values. As for the other tests, we present the results only for the solution of the implicit equations that are obtained at each time step. We use the algorithm of Kutznetsov, which does not contain a coarse model, and in which only *one* iteration is used. Our tests using Chebychev collocation on the local subdomains, indicated that the algorithm is sensitive to the way in which the forcing is decomposed. However, for a partitioning without steep gradients, the algorithm is expected (based on theory) to perform well. The results listed in table 3 are for an application of Kutznetsov's algorithm to the resulting matrix, where a decomposition of the domain (or unknowns in the matrix formulation) is obtained by grouping neighbouring Chebychev nodes for a global discretisation of the problem. No local Chebychev discretisations were used in these tests. Our partition of the unknowns was based on first forming n "subdomains", each consisting of just one unknown (where n is the number of unknowns) and then an extension is formed by including N_o unknowns each from both sides. We list the error in the computed solution, for various choices of time steps Δt, Reynolds numbers Re, convection speeds c, overlap parameter N_o and number of nodes used in the Chebychev discretisation, N. Recall that only one iteration is used in the algorithm. The results indicate that the error decreases as the amount of overlap is increased, untill the error becomes $O(\Delta t)$. It was found that for moderate choices of overlap, the error remained close to the time step, but not smaller. This indicates, that in the matrix version of this algorithm, there are some constraints to the accuracy. And so, it would be preferable to use the algorithm using local Chebychev discretisations and for a smooth partitioning of the forcing. However, it must be mentioned that computational cost of the matrix version of the algorithm was small. For finite difference schemes the two versions of the algorithm,

the matrix version and the domain decomposition version, are equivalent and perform well, see Kutznetsov [19].

7 Extensions to the Navier-Stokes equations.

The Schwarz alternating method is easily generalised and applied to the Stokes equations and the incompressible Navier-Stokes equations, see Lions [20], Fortin and Manouzi [12], and Aboulaich and Fortin [1]. In the case of the Stokes problem, convergence proofs are known, see Lions [20].

The generalisation of the nonlinear Gauss-Seidel-Newton method to the Stokes and Navier-Stokes equations has been done for certain finite difference methods, see Brandt and Dinar [4], and Hackbusch [16]. It is referred to as a Distributive Gauss-Seidel method. The extension to other types of discretisations requires more study.

We intend to study and test Kutznetsov's method for the case of the Stokes and Navier-Stokes equations.

8 Some future plans.

Our studies have led us to focus on various issues, concerning the simulation of turbulence in the incompressible Navier-Stokes equations, in order to reduce the computational cost. Specifically, we would like to study how to adaptively coarsen or refine the mesh in space as the minimum scale of the flow changes in time. This can be estimated quantitatively using the bound for the minimum scale given in Henshaw, Kreiss and Reyna [17] which depends on the Reynolds number of the flow based on the kinematic viscosity and maximum velocity gradients. In addition, local refinement may be possible based on local bounds for the maximum velocity gradients.

For regimes of the flow where large-scale structures are present, we would like to test the use of different time scales for the large-scale and small-scale components of the solution. We intend to study these in the context of the Stokes and Navier-Stokes equations in two dimensions.

Finally, we would also like to focus on turbulent flows which are locally homogeneous, in which case it may be possible to use direct simulation on a few subregions to generate a model for the small-scale features of the solution on the whole domain.

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